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Improved estimates for the approximation numbers of Hardy-type operators

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Abstract

Consider the Hardy-type integral operator $T : L^p(a,b) \to L^p(a,b), -\infty \leqslant a < b \leqslant \infty$, which is defined by

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)\,dt.$$

In the papers by Edmunds et al. (J. London Math. Soc. (2) 37 (1988) 471) and Evans et al. (Studia Math. 130 (2) (1998) 171) upper and lower estimates and asymptotic results were obtained for the approximation numbers $a_n(T)$ of T. In case $p = 2$ for "nice" u and v these results were improved in Edmunds et al. (J. Anal. Math. 85 (2001) 225). In this paper, we extend these results for $1 < p < \infty$ by using a new technique. We will show that under suitable conditions on u and v,

$$\limsup_{n\to\infty} n^{1/2} \left| \lambda_p^{-1/p} \int_a^b |u(t)v(t)|\,dt - n a_n(T) \right|$$

$$\leqslant c(||u'||_{p'/(p'+1)} + ||v'||_{p/(p+1)})(||u||_{p'} + ||v||_p) + 3\alpha_p ||uv||_1,$$

where $||w||_p = (\int_a^b |w(t)|^p\,dt)^{1/p}$ and λ_p is the first eigenvalue of the p-Laplacian eigenvalue problem on $(0,1)$.

Keywords: Approximation numbers; Hardy-type operators; Integral operators; Weighted spaces

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1. Introduction

The approximation numbers $a_n(T)$ of

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt \tag{1}$$

as an operator from $L^p(\mathfrak{R}^+)$ to itself were studied in [EEH1,EEH2,EHL1,EHL2]. Here $\mathfrak{R}^+ = (0,\infty)$, $1<p<\infty$, and u, v are real-valued functions with $u \in L^{p'}_{\text{loc}}(\mathfrak{R}^+)$, and $v \in L^p(\mathfrak{R}^+)$; as usual, $p' = p/(p-1)$.

If T is bounded from $L^p(\mathfrak{R}^+)$ to itself, then to each $\varepsilon>0$ there corresponds $N(\varepsilon)\in \mathbf{N}$ such that

$$a_{N(\varepsilon)+2}(T) \leqslant \frac{\varepsilon}{\sqrt{2}} \leqslant a_{N(\varepsilon)}(T) \tag{2}$$

(see [EEH1]).

Under certain restrictions on u and v it was shown that

$$\lim_{n\to\infty} n a_n(T) = \alpha_p \int_0^\infty |u(t)v(t)|\,dt \tag{3}$$

(see [EEH2] for $p=2$, [EHL2] for $1\leqslant p \leqslant \infty$ and for related results see also [NS].)

In [EHL2] it was shown that (3) is true also for the Hardy-type operator on trees and for $1<p<\infty$.

Further extensions were given in [LL,LMN] to deal with the cases in which T is viewed as a map from L^p to L^q, for any $p, q \in [1,\infty]$.

In the paper [EKL], estimate (3) was improved in the case $p=2$ (L^2 is the Hilbert space and then given any point it is simple to find the nearest element in any closed subspace by using a linear projection, and it is known that $\alpha_p = 1/\pi$). It was shown that under some conditions on u and v we have

$$\limsup_{n\to\infty} n^{1/2}\left| n a_n(T) - \frac{1}{\pi}\int_a^b |uv| \right|$$
$$\leqslant 3\sqrt{2}(\|u'\|_{2/3,I} + \|v'\|_{2/3,I})(\|u\|_{2,I} + \|v\|_{2,I}) + \frac{3}{\pi}\|uv\|_{1,I},$$

I being an arbitrary interval in $\mathfrak{R}$.

In the present paper, we will extend this result to $1<p<\infty$. Under further conditions on u and v we get for the approximation numbers of the map $T: L^p(I) \to L^p(I)$ the following estimates:

$$\limsup_{n\to\infty} n^{1/2}\left| n a_n(T) - \alpha_p\int_a^b |uv| \right|$$
$$\leqslant 3c(p,p')(\|u'\|_{p'/(p'+1),I} + \|v'\|_{p/(p+1),I})(\|u\|_{p',I} + \|v\|_{p,I}) + 3\alpha_p\|uv\|_{1,I},$$

where $\alpha_p = (1/\lambda_p)^{1/p}$ (λ_p is the first eigenvalue of the p-Laplacian problem on (0,1) and $\lambda_p = (\frac{2\pi}{\sin(\pi/p)})^p \frac{1}{p' p^{p-1}}$. Thus,

$$a_n(T) = \frac{\alpha_p}{n} \int_I |u(t)v(t)|dt + O(n^{-3/2}),$$

and under the conditions which we impose, the exponent $-\frac{3}{2}$ cannot be much improved. This is the first theorem of this kind which is covering the case $p \neq 2$ and it is surprising that there is the same power $n^{1/2}$ for any $1 < p < \infty$. We do not know at the moment whether or not it is possible to show the existence of a genuine second term in the expansion of $a_n(T)$. Our results follow from the systematic use of the function A introduced in [EHL1] together with techniques based on those in [EEH2,EKL].

2. Preliminaries

Throughout the paper, we shall assume that $-\infty \leqslant a < b \leqslant \infty$ and that

$$u \in L^{p'}(a,b), \quad v \in L^p(a,b) \quad \text{and} \quad u, v > 0 \ \text{on} \ (a,b). \tag{4}$$

Under these restrictions on u and v it is well known (see, for example, [EEH1, Theorem 1]) that the norm $||T||$ of the operator $T : L^p(a,b) \to L^p(a,b)$ in (1) satisfies

$$||T|| \sim \sup_{x \in (a,b)} ||u\chi_{(a,x)}||_{p',(a,b)} ||v\chi_{(x,b)}||_{p,(a,b)}. \tag{5}$$

Here χ_S denotes the characteristic function of the set S and

$$||f||_{p,I} = \left(\int_I |f(t)|^p \, dt \right)^{1/p}, \quad 1 < p < \infty, \quad I \subset (a,b).$$

Moreover, by $F_1 \sim F_2$ we mean that $C^{-1} F_1 \leqslant F_2 \leqslant C F_1$ for some positive constant $C \geqslant 1$ independent of any variables in $F_1, F_2 \geqslant 0$.

Given any interval $I = (c,d) \subset (a,b)$, define

$$J(I) = \sup_{x \in I} ||u\chi_{(c,x)}||_{p',I} ||v\chi_{(x,d)}||_{p,I}.$$

A straightforward modification of Lemma 2.1 in [EHL1] shows that for any $d \in (a,b)$, the function $J((\cdot,d))$ is continuous and non-increasing on (a,d). Now, for any $x \in I = (c,d) \subset (a,b)$, set

$$(T_I f)(x) = v(x)\chi_I(x) \int_a^x u(t)\chi_I(t) f(t) \, dt.$$

Then the norm of the operator $T_I : L^p(I) \to L^p(I)$ satisfies

$$||T_I|| \sim J(I).$$

We next introduce a function A which will play a key role in the paper. Given $I = (c,d) \subset (a,b)$, set

$$A(I) := \sup_{||f||_{p,I}=1} \inf_{\alpha \in \Re} ||Tf - \alpha v||_{p,I}.$$

From (4) it follows that T is a compact operator from L^p into L^p (see [EGP] or [OK]) and then from [EHL2, Theorem 3.8] we have that

$$A(I) = \inf_{x\in I} ||T_{x,I}|L^p(I)\to L^p(I)||,$$

where

$$T_{x,I}f(\cdot) := v(\cdot)\chi_I(\cdot)\int_x^{\cdot} v(t)\chi_I(t)\,dt.$$

Lemma 2.1. *Let* $I=(c,d)\subset(a,b)$ *and* $1\leqslant p\leqslant\infty$. *Then* $||T_{x,I}|L^p(I)\to L^p(I)||$ *is continuous in* x.

Proof. See Lemma 3.4 in [EHL2] and put $\Gamma=(a,b)$ and $K=I$. $\square$

Lemma 2.2. *Suppose that* u *and* v *satisfy* (4), $a\leqslant c<d\leqslant b$ *and* $1<p<\infty$. *Then*:

(1) *The function* $A(\cdot,d)$ *is non-increasing and continuous on* (a,d).
(2) *The function* $A(c,\cdot)$ *is non-decreasing and continuous on* (c,b).
(3) $\lim_{y\to a_+} A(a,y)=\lim_{y\to b_-} A(y,b)=0$.

Proof. For $p=2$ this lemma was proved in [EKL], (see [EKL, Lemma 2.3]). The proof of this lemma for $p\neq 2$ can be obtained by modification of the proof of Lemma 2.3 from [EKL]. $\square$

Lemma 2.3. *Suppose that* $T: L^p(a,b)\to L^p(a,b)$ *is compact and* $1<p<\infty$. *Let* $I=(c,d)$ *and* $J=(c',d')$ *be subintervals of* (a,b), *with* $J\subset I$, $|J|>0$, $|I-J|>0$, $\int_a^b v^p(x)\,dx<\infty$ *and* $u,v>0$ *on* I. *Then*

$$A(I)>A(J)>0. \tag{6}$$

Proof. For $p=2$ this lemma was proved in [EKL] by using that the projection on the closest element is a linear projection in L^2. This is not true for $p\neq 2$ and in this proof $(1<p<\infty)$ we will use Lemma 3.5 from [EHL2].

Let $0\leqslant f\in L^p(J)$, $0<||f||_{p,J}\leqslant||f||_{p,I}\leqslant 1$ with $\operatorname{supp} f\subset J$. Let $y\in J$, then

$$||T_{(c',y)}||_{p,J}>0 \quad\text{and}\quad ||T_{(y,d')}||_{p,J}>0$$

and then from [EHL2, Lemma 3.5] we have

$$\min\{||T_{(c',y)}||_{p,J}, ||T_{(y,d')}||_{p,J}\}\leqslant \min_{x\in J} ||T_{x,J}||_{p,J},$$

which means $A(J)>0$.

Next, let us suppose that $c=c'<d'<d$. By Evans et al. [EHL2], Theorem 3.8, there exist $x_0\in J$ and $x_1\in I$ such that $A(J)=||T_{x_0,J}||_{p,J}$ and $A(I)=||T_{x_1,I}||_{p,I}$. Since $u,v>0$ on I, it is then quite easy to see that $x_0\in J^o$ and $x_1\in I^o$.

If $x_0 = x_1$, then, since $u, v > 0$ on I, we get

$$A(I) = ||T_{x_1,I}||_{p,I} > ||T_{x_1,I}||_{p,J} = ||T_{x_1,J}||_{p,J} = A(J).$$

If $x_0 \neq x_1$, then

$$A(I) = ||T_{x_1,I}||_{p,I} \geqslant ||T_{x_1,I}||_{p,J} \geqslant ||T_{x_1,J}||_{p,J} > ||T_{x_0,J}||_{p,J} = A(J).$$

The case $c < c' < d' = d$ could be proved similarly and the case $c < c' < d' < d$ follows from previous cases and the monotonicity of $A(I)$. □

Remark 2.4. It follows from the continuity of A that for sufficiently small $\varepsilon > 0$ there is an a_1, $a < a_1 < b$, for which $A(a_1, b) = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $b = a_0 > a_1 > \cdots > a_{N(\varepsilon)} = a$ with $A(a_i, a_{i-1}) = \varepsilon, i = 1, 2, \ldots, N(\varepsilon) - 1$ and $A(a, a_{N(\varepsilon)-1}) \leqslant \varepsilon$.

By the same arguments as in the proof of Lemma 2.6 from [EKL] we have:

Lemma 2.5. *If $T : L^p(a,b) \to L^p(a,b)$ be compact and $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ then the number $N(\varepsilon)$ is a non-increasing function of ε which takes on every sufficiently large integer value.*

The quantity $N(\varepsilon)$ is useful in the derivation of upper and lower estimates for the approximation numbers of T.

Lemma 2.6. *For all $\varepsilon \in (0, ||T||)$,*

$$a_{N(\varepsilon)+2}(T) \leqslant \varepsilon \leqslant a_{N(\varepsilon)+1}(T).$$

Proof. This follows from [EHL2, Lemma 3.19] (put $K = (a,b)$). □

A version of this result, with a slightly different $N(\varepsilon)$, was first proved in [EEH1] and was then extended in [EHL1]. For general u and v it is impossible to find a simple relation between ε and $N(\varepsilon)$, but by using the properties of A the behavior of $\varepsilon N(\varepsilon)$ when $\varepsilon \to 0_+$ can be determined.

Lemma 2.7. *Given $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ we have*

$$\lim_{\varepsilon \to 0_+} \varepsilon N(\varepsilon) = \alpha_p \int_a^b |u(t)v(t)|\, dt.$$

This result follows from an adaptation of the argument of [EHL2]; see, in particular, Theorem 6.4 of that paper. Together with Lemma 2.6 this shows, again using the techniques of [EHL2], that the following theorem holds.

Theorem 2.8. *Given* $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ *the operator* T *defined in* (1) *satisfies*

$$\lim_{n\to\infty} n a_n(T) = \alpha_p \int_a^b |u(t)v(t)|\, dt,$$

where $\alpha_p = A((0,1),1,1)$.

A result of this type was established under weaker conditions on u and v in [EHL2].

In [EL] was showed that $A((0,1),1,1)$ is equal to $(1/\lambda_p)^{1/p}$ where $\lambda_p = (\frac{2\pi}{\sin(\pi/p)})^p \frac{1}{p' p^{p-1}}$ is the first eigenvalue of the p-Laplacian problem on $(0,1)$.

3. Technical results

Here, we give some results of a technical nature which will prove very useful in the sequel. We begin with some facts about the function A which were proved in [EHL2] (see Lemmas 4.1, 4.2 and 4.3 in [EHL2] with $\Gamma = I$) for the Hardy-type operators on trees.

Lemma 3.1. (i) *Let* $I = (c,d) \subseteq (a,b)$ *and suppose that* u *and* v *are constant functions over* I. *Then*

$$A(I,u,v) = |I||u||v| A((0,1),1,1).$$

(ii) *Let* $I = (c,d) \subset (a,b)$ *and suppose that* $v \in L^p(I)$ *and* $u_1, u_2 \in L^{p'}(I)$. *Then*

$$|A(I,u_1,v) - A(I,u_2,v)| \leqslant ||u_1 - u_2||_{p',I} ||v||_{p,I}.$$

(iii) *Let* $I = (c,d) \subset (a,b)$ *and suppose that* $u \in L^{p'}(I)$ *and* $v_1, v_2 \in L^p(I)$. *Then*

$$|A(I,u,v_1) - A(I,u,v_2)| \leqslant 2||u||_{p',I} ||v_1 - v_2||_{p,I}.$$

We now turn to the approximation of functions from L^p and $L^{p'}$ by step-functions.

Suppose $u \in L^{p'}(a,b)$ and $v \in L^p(a,b)$ and let $\alpha > 0$. We define $m_\alpha \in \mathbf{N}$ by the following requirements:

There exist two step-functions, u_α and v_α, each with m_α steps, say,

$$u_\alpha(x) := \sum_{j=1}^{m_\alpha} \xi_j \chi_{w_\alpha(j)}(x), \qquad v_\alpha(x) := \sum_{j=1}^{m_\alpha} \psi_j \chi_{w_\alpha(j)}(x), \tag{7}$$

where $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a family of non-overlapping intervals covering (a,b), such that for

$$\alpha_u := ||u - u_\alpha||_{p',(a,b)} \qquad \text{and} \qquad \alpha_v := ||v - v_\alpha||_{p,(a,b)},$$

we have

$$\text{(i)} \qquad \max(\alpha_u, \alpha_v) \leqslant \alpha, \tag{8}$$

and

(ii) for any step-functions u'_α, v'_α with less than m_α steps, say n_α steps, $n_\alpha < m_\alpha$, $\max(||u - u'_\alpha||_{p',(a,b)}, ||v - v'_\alpha||_{p,(a,b)}) > \alpha$.

Thus, m_α is the minimum number of steps needed to approximate u in $L^{p'}$ and v in L^p with the required accuracy. Note that, plainly,

$$||u - u_\alpha||_{p',(a,b)} \leqslant \alpha, \qquad ||v - v_\alpha||_{p,(a,b)} \leqslant \alpha.$$

The best way to choose ξ_i and ψ_i for given $\{w_\alpha\}_{j=1}^{m_\alpha}$ is by finding ξ_i and ψ_i such that

$$\int_{w_\alpha(i)} |u(t) - \xi_i|^{p'-1} \operatorname{sgn}(u(t) - \xi_i)\, dt = 0$$

and

$$\int_{w_\alpha(i)} |v(t) - \psi_i|^{p-1} \operatorname{sgn}(v(t) - \psi_i)\, dt = 0$$

(see [S, Theorem 1.11]).

It turns out that the relationship between α and m_α is crucial for us; we next address this matter.

Lemma 3.2. *Suppose $u \in C(a,b) \cap L^{p'}(a,b)$ and $v \in C(a,b) \cap L^p(a,b)$, at least one of them, say u, being non-constant. Then, when α decreases to 0, m_α increases to ∞.*

Proof. This lemma was proved in the case $p = 2$ in [EKL], (see [EKL, Lemma 3.4]). The proof for the case $p = 2$ from [EKL] can be simply modified for $1 < p < \infty$. □

Lemma 3.3. *Suppose $u \in C(a,b) \cap L^{p'}(a,b)$ and $v \in C(a,b) \cap L^p(a,b)$, at least one of them, say u, being non-constant. Fix $\alpha > 0$ and set $\Lambda_\alpha = \{\beta; 0 < \beta \leqslant \alpha \text{ and } m_\beta = m_\alpha\}$. Then, Λ_α is an interval with $\gamma = \inf \Lambda_\alpha$ and $\gamma \in \Lambda_\alpha$.*

Proof. By straightforward modification of the proof of Lemma 3.5, [EKL] for the case $p = 2$ we can get the proof for $1 < p < \infty$. □

Lemma 3.4. *Suppose that $u \in L^{p'}(a,b) \cap C(a,b)$ and $v \in L^p(a,b) \cap C(a,b)$ are not equal to zero on (a,b), indeed, assume at least one of u and v be non-constant on (a,b). Then, there exists $\alpha_0 > 0$ such that given any $\alpha, 0 < \alpha < \alpha_0$, there exists a $\beta, 0 < \beta < \alpha$, with $m_\beta = m_\alpha + 1$ or $m_\beta = m_\alpha + 2$.*

Proof. By simple modification of the proof of Lemma 3.6 [EKL] for the case $p = 2$ we can get the proof for $1 < p < \infty$. □

Lemma 3.5. *Let $-\infty \leqslant a < b \leqslant \infty$ and suppose that $u' \in L^{p'/(p'+1)}(a,b) \cap C(a,b)$. For each small $h > 0$ define*

$$x_1 = -\frac{1}{h}, \quad x_{i+1} := x_i + h \quad \text{for } i \in 1, \ldots, [2/h^2];$$

put $J_i = (a,b) \cap (x_i, x_{i+1})$, $i \in 1, \ldots, [2/h^2]$.

Then

$$\int_a^b |u'(t)|^{p'/(p'+1)}\,dt = \lim_{h\to 0}\sum_{i=1}^{[2/h^2]} |J_i| \max_{x\in J_i} |u'(x)|^{p'/(p'+1)}$$
$$= \lim_{h\to 0}\sum_{j=1}^{[2/h^2]} |J_i| \min_{x\in J_i} |u'(x)|^{p'/(p'+1)}.$$

Proof. Simply use the definition of the integral. □

We are now prepared to establish an important estimate for $\limsup_{\alpha\to 0_+} \alpha m_\alpha$.

Theorem 3.6. *Suppose* $u\in L^{p'}(a,b)$, $v\in L^p(a,b)$ *and* $u'\in L^{p'/(p'+1)}(a,b)\cap C(a,b)$, $v'\in L^{p/(p+1)}(a,b)\cap C(a,b)$. *Then,*

$$\limsup_{\alpha\to 0_+} \alpha m_\alpha \leqslant c(p,p')(||u'||_{p'/(p'+1),(a,b)} + ||v'||_{p/(p+1),(a,b)}).$$

Proof. This theorem was proved for $p=2$ in [EKL]. With help of our previous lemmas it is simple to modify the proof of Theorem 3.8 [EKL] for the case $1<p<\infty$. □

4. The Main theorem

The next theorem give us quite precise information about remainder estimates for $N(\varepsilon)$.

Theorem 4.1. *Let* $-\infty\leqslant a<b\leqslant\infty$, *let* $u\in L^{p'}(a,b)$, $v\in L^p(a,b)$ *and suppose that* $u'\in L^{p'/(p'+1)}(a,b)\cap C([a,b])$, $v'\in L^{p/(p+1)}(a,b)\cap C([a,b])$. *Then*

$$\limsup_{\varepsilon\to 0_+}\left|\alpha_p\int_a^b |u(t)v(t)|\,dt - \varepsilon N(\varepsilon)\right| N^{1/2}(\varepsilon)$$
$$\leqslant c(p,p')(||u'||_{p'/(p'+1),(a,b)} + ||v'||_{p/(p+1),(a,b)})\left(||u||_{p',(a,b)} + ||v||_{p,(a,b)}\right)$$
$$+ 3\alpha_p||uv||_{1,(a,b)},$$

where $\alpha_p = A((0,1),1,1)$ *and* $c(p,p')$ *is a constant depending only on* p *and* p'.

Proof. For $p=2$ this theorem was proved in [EKL]. The proof of this key theorem for $1<p<\infty$ can be obtained by easy modification of the proof of Theorem 4.1 from [EKL]. □

From the previous theorem follows the Main theorem with the estimate for the approximation numbers of T given by (1).

Theorem 4.2. *Let* $-\infty \leqslant a < b \leqslant \infty$, *suppose that* $u \in L^{p'}(a,b)$, $v \in L^p(a,b)$ *and let* $u' \in L^{p'/(p'+1)}(a,b) \cap C((a,b))$, $v' \in L^{p/(p+1)}(a,b) \cap C((a,b))$. *Then*

$$\limsup_{n\to\infty} n^{1/2} \left| \alpha_p \int_a^b |uv|\, dt - na_n \right|$$

$$\leqslant 3\alpha_p \int_a^b |uv|\, dt$$

$$+ 3c(p,p')(||u'||_{p'/(p'+1),(a,b)} + ||v'|)||_{p/(p+1),(a,b)})(||u||_{p',(a,b)} + ||v||_{p,(a,b)}),$$

where $\alpha_p = \frac{\sin(\pi/p)}{2\pi} p' p^{p-1}$.

Proof. Simply use Theorem 4.1, Lemmas 2.5, 2.6 and the fact that $\lim_{n\to\infty} n^{\alpha} a_n(T) = 0$ for any $\alpha < 1$ and $\alpha_p = (\frac{1}{\lambda_p})^{1/p}$. $\square$

For a bounded interval (a,b), it follows immediately from Hölder's inequality that Theorem 4.2 gives rise to

Theorem 4.3. *Let* $-\infty < a < b < \infty$ *and suppose that* $u', v' \in C([a,b])$. *Then*

$$\limsup_{n\to\infty} n^{1/2} |\alpha_p \int_a^b |uv|\, dt - na_n|$$

$$\leqslant 3\alpha_p \int_a^b |uv|\, dt + 3c(p,p')(b-a)$$

$$\times \quad (||u'||_{p',(a,b)} + ||v'||_{p,(a,b)})(||u||_{p',(a,b)} + ||v||_{p,(a,b)}),$$

where $\alpha_p = \frac{\sin(\pi/p)}{2\pi} p' p^{p-1}$.

From the following observation we can see that any optimal exponent from Theorem 4.2 has to belong to $[\frac{1}{2}, 1]$.

Observation 4.4. *Let* $-\infty \leqslant a < b \leqslant \infty$.

(i) *Let* $\alpha < 1/2$. *Then for every* $u \in L^{p'}(a,b)$, $v \in L^p(a,b)$ *with* $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b])$, $v' \in L^{p/(p+1)}(a,b) \cap C([a,b])$ *we have*

$$\limsup_{n\to\infty} n^{\alpha} \left| \alpha_p \int_a^b |uv| dt - na_n(T) \right| = 0.$$

(ii) *Let* $\alpha > 1$. *Then there exist a and b, and functions u and v satisfying the conditions of Theorem* 4.2 *on the interval defined by a and b, such that*

$$\limsup_{n\to\infty} n^{\alpha} \left| \alpha_p \int_a^b |uv| dt - na_n(T) \right| = \infty.$$

Proof. (i) follows from the proof of Theorem 4.1. Put $m_\alpha = [N^\alpha(\varepsilon)]$ or $[N^\alpha(\varepsilon)] + 1$.

(ii) Take $(a,b) = (0,1)$ and $u = 1,\ v = 1 + x$. Then put $m_\alpha = [N^\alpha(\varepsilon)]$ in the proof of Theorem 4.1 and the result follows. □

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